

NORMALITY CRITERIA CONCERNING COMPOSITE MEROMORPHIC FUNCTIONS

VIRENDER SINGH AND KULDEEP SINGH CHARAK

ABSTRACT. In this paper, we prove normality criteria for families of meromorphic functions involving sharing of a holomorphic function by a certain class of differential polynomials. Results in this paper extend the works of different authors carried out in recent years.

1. Introduction and Main Results

Let \mathcal{F} be a family of meromorphic functions in a domain D with all zeros of multiplicity at least k , $P_w \equiv P(z, w) := \prod_{l=1}^n (w - a_l(z))$ be a polynomial with holomorphic functions $a_l(z)$ ($1 \leq l \leq n$) as the coefficients, $\alpha(z)$ be a holomorphic function on D such that $P(z_0, w) - \alpha(z_0)$ has at least two distinct zeros for every $z_0 \in D$; and

$$M[f] := f^{n_0} (f')^{n_1} \dots (f^{(k)})^{n_k}, \quad (k \geq 1),$$

be a differential monomial of $f \in \mathcal{F}$ with degree $\gamma_M := \sum_{j=0}^k n_j$ and weight $\Gamma_M := \sum_{j=0}^k (j+1)n_j$, where $n_0 \geq 1$ and n_j ($1 \leq j \leq k$) are non-negative integers such that $\sum_{j=1}^k n_j \geq 1$.

Regarding the normality of \mathcal{F} , we consider the following question:

Question 1.1. If for every $f, g \in \mathcal{F}$, $P_w \circ M[f]$ and $P_w \circ M[g]$ share $\alpha(z)$ IM, is it true that \mathcal{F} is a normal family?

The motivation for proposing this question is by the works of Fang and Yuan [3], Bergweiler [1], Yuan, Li and Xiao [14] and Yuan, Xiao and Wu ([15],[12]). In fact, Fang and Yuan [3, Theorem 4, p.323] proved: *If \mathcal{G} is a family of holomorphic functions in a domain D , $P(z)$ is a polynomial of degree at least 2, $\alpha(z)$ is a holomorphic function such that $P(z) - \alpha(z)$ has at least two distinct zeros, and if $P \circ f(z) \neq \alpha(z)$ for each $f \in \mathcal{G}$, then \mathcal{G} is normal in D .* Bergweiler [1, Theorem 4, p.654] generalised this result and proved: *If \mathcal{G} is a family of holomorphic functions in a domain D , $R(z)$ is a rational function of degree at least 2, $\alpha(z)$ is a non constant meromorphic function, and if $R \circ f(z) \neq \alpha(z)$ for*

2010 *Mathematics Subject Classification.* 30D45, 30D35.

Keywords and phrases. Normal families, Meromorphic function, Differential monomial, Shared function.

The research work of the first author is supported by the CSIR India.

each $f \in \mathcal{G}$, then \mathcal{G} is normal in D . Yuan, Li and Xiao [14] further improved this result and proved: *If $\alpha(z)$ is a non constant meromorphic function, $R(z)$ is a rational function of degree at least 2, and $R \circ f$ and $R \circ g$ share $\alpha(z)$ IM for all $f(z), g(z) \in \mathcal{G}$, then \mathcal{G} is normal in D if one of the following conditions holds:*

- (1) $R(z) - \alpha(z_0)$ has at least two distinct zeros or poles for any $z_0 \in D$
- (2) There exists $z_0 \in D$ such that $R(z) - \alpha(z_0) := P(z)/Q(z)$ has only one distinct zero (or pole) β_0 and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0 , respectively, satisfy $k \neq lp$ (or $k \neq lq$), for each $f \in \mathcal{F}$, where P and Q are two co-prime polynomials with degree p and q respectively.

And in 2011, Yuan, Xiao and Wu ([15], [12]) extended this result and proved:

Theorem A [15, Theorem 1.4, p.436]. Let $\alpha(z)$ be a holomorphic function, \mathcal{G} be a family of meromorphic functions in a domain D and $P(z)$ be a polynomial of degree at least 3. If $P \circ f(z)$ and $P \circ g(z)$ share $\alpha(z)$ IM for each pair $f, g \in \mathcal{G}$ and one of the following conditions holds:

- (1) $P(z) - \alpha(z_0)$ has at least three distinct zeros for any $z_0 \in D$
- (2) There exists $z_0 \in D$ such that $P(z) - \alpha(z_0)$ has at most two distinct zeros and $\alpha(z)$ is non constant. Assume that β_0 is the zero of $P(z) - \alpha(z_0)$ with multiplicity p and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0 , respectively, satisfy $k \neq lp$, for all $f \in \mathcal{G}$,

then \mathcal{G} is normal in D .

Theorem B [12, Theorem 1.2, p.2]. Let $\alpha(z)$ be a holomorphic function and \mathcal{G} a family of holomorphic functions in a domain D . If $P_w \circ f(z)$ and $P_w \circ g(z)$ share $\alpha(z)$ IM for each pair $f, g \in \mathcal{G}$ and one of the following conditions holds:

- (1) $P(z_0, w) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$
- (2) There exists $z_0 \in D$ such that $P(z_0, w) - \alpha(z_0)$ has only one distinct zero and $\alpha(z)$ is non constant. Assume that β_0 is the zero of $P(z_0, w) - \alpha(z_0)$ and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0 , respectively, satisfy $k \neq lp$, for all $f \in \mathcal{G}$,

then \mathcal{G} is normal in D .

In this paper we answer Question 1.1 as

Theorem 1.2. *If for every $f, g \in \mathcal{F}$, $P_w \circ M[f]$ and $P_w \circ M[g]$ share $\alpha(z)$ IM, then \mathcal{F} is a normal family in D .*

Example 1.3. Consider the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$, where

$$f_n(z) = nz^{k-1}; k \geq 2$$

in the unit disk \mathbb{D} , $P(z, w) = (w + e^z)(w - e^z)$ and $\alpha(z) \equiv 0$. Then, for every $f_n, f_m \in \mathcal{F}$, $P_w \circ M[f_n](z)$ and $P_w \circ M[f_m](z)$ share $\alpha(z)$ IM. However, the family \mathcal{F} is not normal in \mathbb{D} . Thus, the condition that every $f \in \mathcal{F}$ has zeros of multiplicity at least k is essential in Theorem 1.2.

Example 1.4. Consider the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$, where

$$f_n(z) = \frac{1}{nz}$$

in the unit disk \mathbb{D} , $P(z, w) = (w + iz e^z)(w - iz e^z)$ and $\alpha(z) = z^2 e^{2z}$. Then

$$P_w \circ M[f_n](z) = \frac{\prod_{r=1}^k (r!)^{2n_r}}{n^{2\gamma_M} z^{2\Gamma_M}} + z^2 e^{2z}$$

Clearly, for every $f_n, f_m \in \mathcal{F}$, $P_w \circ M[f_n](z)$ and $P_w \circ M[f_m](z)$ share $\alpha(z)$ IM. However, the family \mathcal{F} is not normal in \mathbb{D} . Thus, the condition that $P(z_0, w) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$ is essential in Theorem 1.2.

Let $S = \{a_i(z)/i = 1, 2, \dots, n\}$ be the set of holomorphic functions. Then we say that two meromorphic functions f and g share the set S IM if $\overline{E}_f(S) = \overline{E}_g(S)$, where $\overline{E}_\phi(S) = \{z \in D / \prod_{i=1}^n (\phi(z) - a_i(z)) = 0\}$. In that case we write $f(z) \in S \Leftrightarrow g(z) \in S$.

From Theorem 1.2, we immediately obtain the following corollary by setting $\alpha(z) \equiv 0$ and $P(z, w)$ a polynomial in variable w that vanishes exactly on a finite set of holomorphic functions. This result can be seen as an extension of a result due to Fang and Zalcman [4] to a general class of differential monomials.

Corollary 1.5. *Let S be a finite set of holomorphic functions with at least 2 elements. If, for every $f, g \in \mathcal{F}$ and for $z \in D$,*

$$M[f](z) \in S \Leftrightarrow M[g](z) \in S,$$

then \mathcal{F} is normal in D .

Further, one can ask what can be said about normality of family \mathcal{F} if set S in Corollary 1.5 has cardinality equal to one. In this direction, we prove the following result which in turn extends the results of Yunbo and Zongsheng [13], Meng and Hu [6], Charak and Sharma [2] and Sun [10].

Theorem 1.6. *Let $n_0, n_1, n_2, \dots, n_k, k$ be the nonnegative integers such that $n_0 \geq 2, n_k \geq 1$ and $k \geq 1$. Let \mathcal{G} be a family of meromorphic functions in a domain D and $\alpha(z) \not\equiv 0$ be a polynomial of degree p . Suppose that each $f \in \mathcal{G}$ has zeros of multiplicity at least $k + p + 1$ and poles of multiplicity at least $p + 1$. If, for every $f, g \in \mathcal{G}$, $M[f](z)$ and $M[g](z)$ share $\alpha(z)$ IM, then \mathcal{G} is normal in D .*

In [3, Theorem 4, p.323], for any $f \in \mathcal{G}$, $P \circ f(z) \neq \alpha(z)$ for every $z \in D$, then it is natural to investigate the case when $P \circ f(z) - \alpha(z)$ has zeros. In this direction, we have the following results:

Theorem 1.7. *If for every $f \in \mathcal{F}$, $P_w \circ M[f](z) - \alpha(z)$ has at most one zero, then \mathcal{F} is normal in D .*

Theorem 1.8. *If for every $f \in \mathcal{F}$, $P_w \circ M[f](z) = \alpha(z)$ implies $|f(z)| \geq M$, for some $M > 0$, then \mathcal{F} is normal in D .*

Based on ideas from [2], one may extend Theorem 1.2 under the weaker hypothesis of partial sharing of holomorphic function in the following way.

Theorem 1.9. *If for every $f \in \mathcal{F}$, there exist $g \in \mathcal{F}$ such that $P_w \circ M[f](z)$ share $\alpha(z)$ partially with $P_w \circ M[g](z)$, then \mathcal{F} is normal in D provided $P_w \circ M[g](z) \not\equiv \alpha(z)$.*

2. Proofs of Main Results

Besides Zalcman's lemma [16, p.216], the proofs of our main results rely on the following value distribution results:

Lemma 2.1. *Let $n_0, n_1, n_2, \dots, n_k, k, p$ be the non-negative integers with $n_0 \geq 2, \sum_{j=1}^k n_j \geq 1$ and $k \geq 1$. Let $\alpha(z) \not\equiv 0$ be a polynomial of degree p and f be a non constant rational function having zeros of multiplicity at least $k + p$ and poles of multiplicity at least $p + 1$. Then $f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k} - \alpha(z)$ has at least two distinct zeros.*

Proof. Let $\Psi := f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}$. Suppose on contrary that $\Psi(z) - \alpha(z)$ has at most one zero. We distinguish the following cases:

Case I: When f is a non constant polynomial. Since f has only zeros of multiplicity at least $k + p$ and $\alpha(z)$ is a polynomial of degree p , we can see that $\Psi(z) - \alpha(z)$ has at least one zero. We set

$$(2.1) \quad \Psi(z) - \alpha(z) = a(z - z_0)^m,$$

where a is a non zero constant and $m > p + 1$ is a positive integer. Then

$$\Psi^{(p+1)}(z) = am(m-1)(m-2) \dots (m-p)(z - z_0)^{m-p-1}$$

implies that z_0 is the only zero of $\Psi^{(p+1)}(z)$. Since f is a non constant polynomial, it follows that z_0 is a zero of $f(z)$ and hence a zero of $\Psi(z)$ of multiplicity at least $\gamma_\Psi(k + p + 1) - \Gamma_\Psi$. Thus $\Psi^{(p)}(z_0) = 0$. But, from (2.1), we have $\Psi^{(p)}(z_0) = \alpha^{(p)}(z_0) \neq 0$, a contradiction.

Case II: When f is a rational but not a polynomial. We set

$$(2.2) \quad f(z) = A \frac{\prod_{i=1}^s (z - a_i)^{m_i}}{\prod_{j=1}^t (z - b_j)^{l_j}},$$

where A is a non-zero constant, $m_i \geq k + p$ ($i = 1, 2, \dots, s$) and $l_j \geq p + 1$ ($j = 1, 2, \dots, t$). Put

$$M = \sum_{i=1}^s m_i$$

and

$$N = \sum_{j=1}^t l_j.$$

Then $M \geq (k + p)s$ and $N \geq (p + 1)t$.

Now,

$$(2.3) \quad \Psi(z) = A^{\gamma_\Psi} \frac{\prod_{i=1}^s (z - a_i)^{(m_i+1)\gamma_\Psi - \Gamma_\Psi}}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi}} g_0(z),$$

where $g_0(z)$ is a polynomial such that $\deg(g_0(z)) \leq (s+t-1)(\Gamma_\Psi - \gamma_\Psi)$.
On differentiating (2.3), we have

$$(2.4) \quad \Psi^{(p)}(z) = A^{\gamma_\Psi} \frac{\prod_{i=1}^s (z - a_i)^{(m_i+1)\gamma_\Psi - \Gamma_\Psi - p}}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi + p}} g_1(z),$$

$$(2.5) \quad \Psi^{(p+1)}(z) = A^{\gamma_\Psi} \frac{\prod_{i=1}^s (z - a_i)^{(m_i+1)\gamma_\Psi - \Gamma_\Psi - p - 1}}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi + p + 1}} g_2(z),$$

where $g_1(z)$ and $g_2(z)$ are the polynomials such that $\deg(g_1(z)) \leq (s+t-1)(\Gamma_\Psi - \gamma_\Psi + p)$ and $\deg(g_2(z)) \leq (s+t-1)(\Gamma_\Psi - \gamma_\Psi + p + 1)$.

Case 2.1 Suppose that $\Psi(z) - \alpha(z)$ has exactly one zero say z_0 . We set

$$(2.6) \quad \Psi(z) = \alpha(z) + \frac{(z - z_0)^l}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi}},$$

where l is a positive integer.

On differentiating (2.6), we have

$$(2.7) \quad \Psi^{(p)}(z) = B + \frac{(z - z_0)^{l-p}}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi + p}} g_3(z),$$

$$(2.8) \quad \Psi^{(p+1)}(z) = \frac{(z - z_0)^{l-p-1}}{\prod_{j=1}^t (z - b_j)^{(l_j-1)\gamma_\Psi + \Gamma_\Psi + p + 1}} g_4(z),$$

where B is a non zero constant and $g_3(z)$, $g_4(z)$ are the polynomials such that $\deg(g_3(z)) \leq pt$ and $\deg(g_4(z)) \leq (p+1)t$. On comparing (2.4) and (2.7), we conclude that $z_0 \neq a_i (i = 1, 2, \dots, s)$.

Case 2.2.1 $l \neq (N-t)\gamma_\Psi + t\Gamma_\Psi + p$

From (2.3) and (2.6), we deduce

$$\begin{aligned} (N-t)\gamma_\Psi + t\Gamma_\Psi &\leq (M+s)\gamma_\Psi - s\Gamma_\Psi + \deg(g_0(z)) \\ &\leq (M+s)\gamma_\Psi - s\Gamma_\Psi + (s+t-1)(\Gamma_\Psi - \gamma_\Psi) \\ &\Rightarrow N < M. \end{aligned}$$

Also, from (2.5) and (2.8), we have

$$\begin{aligned} (M+s)\gamma_\Psi - s\Gamma_\Psi - (p+1)s &\leq \deg(g_4(z)) \leq (p+1)t \leq N \\ \Rightarrow M\gamma_\Psi &\leq s(\Gamma_\Psi - \gamma_\Psi + p + 1) + N \\ \Rightarrow M &< \left(\frac{\Gamma_\Psi - \gamma_\Psi + p + 1}{(k+p)\gamma_\Psi} + \frac{1}{\gamma_\Psi} \right) M \\ \Rightarrow M &< M, \end{aligned}$$

which is absurd.

Case 2.2.2 $l = (N - t)\gamma_\Psi + t\Gamma_\Psi + p$

When $M > N$, then by proceeding similar way as in case 2.2.1, we get a contradiction.

When $M \leq N$, then from (2.5) and (2.8), we have

$$\begin{aligned}
l - p - 1 &\leq \deg(g_2(z)) \leq (s + t - 1)(\Gamma_\Psi - \gamma_\Psi + p + 1) \\
\Rightarrow l &\leq (s + t - 1)(\Gamma_\Psi - \gamma_\Psi + p + 1) + p + 1 \\
\Rightarrow (N - t)\gamma_\Psi + t\Gamma_\Psi + p &\leq (s + t - 1)(\Gamma_\Psi - \gamma_\Psi + p + 1) + p + 1 \\
\Rightarrow N &< M \frac{\Gamma_\Psi - \gamma_\Psi + p + 1}{(k + p)\gamma_\Psi} + N \frac{1}{\gamma_\Psi} \\
\Rightarrow N &< \left(\frac{\Gamma_\Psi - \gamma_\Psi + p + 1}{(k + p)\gamma_\Psi} + \frac{1}{\gamma_\Psi} \right) N \\
\Rightarrow N &< N,
\end{aligned}$$

which is absurd.

Case 2.2 Suppose that $\Psi(z) - \alpha(z)$ has no zero, then by proceeding the same manner as in case 2.1, we get a contradiction. Hence the Lemma follows. \blacksquare

Lemma 2.2. *Let $n_0, n_1, n_2, \dots, n_k, k$ be the non negative integers with $n_0 \geq 2, \sum_{j=1}^k n_j \geq 1$ and $k \geq 1$. Let f be a transcendental meromorphic function such that f has only zeros of multiplicity at least $k + 1$. Then $f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k} - a(z)$ has infinitely many zeros for any small function $a(z) (\neq 0, \infty)$ of f .*

Proof. Let $\Psi := f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}$. Suppose on contrary that $\Psi(z) - a(z)$ has only finitely many zeros. Then by second fundamental theorem of Nevanlinna for three small functions, we have

$$\begin{aligned}
T(r, \Psi) &\leq \overline{N}(r, \Psi) + \overline{N}\left(r, \frac{1}{\Psi}\right) + \overline{N}\left(r, \frac{1}{\Psi - a(z)}\right) \\
(2.9) \quad &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{\Psi}\right) + S(r, \Psi).
\end{aligned}$$

Also, from [2, Theorem 1.1, see (2.2)], we have

$$(2.10) \quad \overline{N}\left(r, \frac{1}{\Psi}\right) \leq \left(1 + \sum_{j=1}^k\right) \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k j \overline{N}(r, f) + S(r, \Psi),$$

where the summation runs over the terms $f^{(j)}$ ($1 \leq j \leq k$) in the monomial Ψ of f . Since the zeros of f has multiplicity at least $k + 1$, we have

$$\begin{aligned}
(2.11) \quad \overline{N}\left(r, \frac{1}{f}\right) &= \overline{N}_{(k+1)}\left(r, \frac{1}{f}\right) \\
&\leq \frac{1}{kn_0 + \gamma - 1} \left(N\left(r, \frac{1}{\Psi}\right) - \overline{N}\left(r, \frac{1}{\Psi}\right) \right).
\end{aligned}$$

Also, one can see that

$$(2.12) \quad \overline{N}(r, f) \leq \frac{1}{\Gamma_\Psi} N(r, \Psi).$$

On combining (2.9), (2.10), (2.11) and (2.12), we get

$$\begin{aligned}
 T(r, \Psi) &\leq \frac{1 + \sum_{j=1}^k j}{kn_0 + \gamma + \sum_{j=1}^k j} N\left(r, \frac{1}{\Psi}\right) + \left(1 + \frac{\sum_{j=1}^k j(kn_0 + \gamma - 1)}{kn_0 + \gamma + \sum_{j=1}^k j}\right) \overline{N}(r, \Psi) + S(r, \Psi) \\
 &\Rightarrow \left(1 - \frac{1}{\Gamma_\Psi} - \frac{1 + \sum_{j=1}^k j(kn_0 + \gamma - 1)}{(kn_0 + \gamma + \sum_{j=1}^k j)\Gamma_\Psi}\right) T(r, \Psi) \leq S(r, \Psi) \\
 &\Rightarrow T(r, \Psi) \leq S(r, \Psi).
 \end{aligned}$$

Thus, by using the inequality [9]

$$T(r, f) + S(r, f) \leq CT(r, \Psi) + S(r, \Psi),$$

where C is a constant and $S(r, f) = S(r, \Psi)$, we get a contradiction. Hence the Lemma follows. \blacksquare

Since normality is local property, we shall always assume that $D = \mathbb{D}$, the open unit disk and hence the point at which the family is assumed to be not normal is the origin.

Proof of Theorem 1.2. Suppose that h_1 and h_2 are two distinct zeros of $P(z_0, w) - \alpha(z_0)$ for any $z_0 \in D$. Suppose on contrary that \mathcal{F} is not normal at the origin. Then by Zalcman's lemma [16, p.216], we can find a sequence $\{f_j\}$ in \mathcal{F} , a sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ such that

$$g_j(\zeta) = \rho_j^{-\beta} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a non constant meromorphic function $g(\zeta)$ on \mathbb{C} having bounded spherical derivative.

We take $\beta = \frac{\Gamma_M}{\gamma_M} - 1$.

Hence,

$$P_w \circ M[f_j](z_j + \rho_j \zeta) - \alpha(z_j + \rho_j \zeta) \rightarrow P_w \circ M[g](\zeta) - \alpha(0).$$

We claim that $P_w \circ M[g](\zeta) - \alpha(0)$ has at least two distinct zeros. We distinguish the following cases:

Case I: Suppose that $P_w \circ M[g](\zeta) - \alpha(0)$ has exactly one zero say a . That is, $P_w \circ M[g](a) = \alpha(0)$ and $P_w \circ M[g](\zeta) \neq \alpha(0)$ for $\zeta \neq a$. Without loss of generality we may assume that $M[g](a) = h_1$. Then $M[g](\zeta) \neq h_2$ and $M[g](\zeta) \neq h_1$ for $\zeta \neq a$. If $g(\zeta)$ is a transcendental meromorphic function, then we have

$$\begin{aligned}
 \Gamma_M \overline{N}(r, g) &\leq N(r, M[g]) \\
 &\leq T(r, M[g]) \\
 &\leq \overline{N}(r, M[g]) + \overline{N}\left(r, \frac{1}{M[g] - h_1}\right) + \overline{N}\left(r, \frac{1}{M[g] - h_2}\right) + S(r, M[g]) \\
 &\leq \overline{N}(r, g) + S(r, g) \\
 &\Rightarrow (\Gamma_M - 1) \overline{N}(r, g) \leq S(r, g)
 \end{aligned}$$

$$(2.13) \quad \Rightarrow \overline{N}(r, g) = S(r, g).$$

Since the zeros of g has multiplicity at least k , we have

$$(2.14) \quad \overline{N}\left(r, \frac{1}{g}\right) \leq \frac{1}{k}N\left(r, \frac{1}{g}\right)$$

Now,

$$\begin{aligned} \gamma_M T(r, g) &= T(r, g^{\gamma_M}) \\ &= T\left(r, \frac{1}{g^{\gamma_M}}\right) + O(1) \\ &\leq T\left(r, \frac{M[g]}{g^{\gamma_M}}\right) + T\left(r, \frac{1}{M[g]}\right) + S(r, g) \\ &\leq T\left(r, \frac{g^{n_0}(g')^{n_1} \dots (g^{(k)})^{n_k}}{g^{n_0}g^{n_1} \dots g^{n_k}}\right) + S(r, g) \\ &= m\left(r, \frac{(g')^{n_1} \dots (g^{(k)})^{n_k}}{g^{n_1} \dots g^{n_k}}\right) + N\left(r, \frac{(g')^{n_1} \dots (g^{(k)})^{n_k}}{g^{n_1} \dots g^{n_k}}\right) + S(r, g) \\ &\leq \sum_{i=1}^k \left[m\left(r, \left(\frac{g^{(i)}}{g}\right)^{n_i}\right) + n_i N\left(r, \left(\frac{g^{(i)}}{g}\right)\right) \right] + S(r, g) \\ &\leq \sum_{i=1}^k i n_i \left[\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) \right] + S(r, g). \end{aligned}$$

That is,

$$(2.15) \quad \gamma_M T(r, g) \leq (\Gamma_M - \gamma_M) \left[\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) \right] + S(r, g).$$

Substituting (2.13) and (2.14) in (2.15), we get

$$\begin{aligned} \gamma_M T(r, g) &\leq \frac{\Gamma_M - \gamma_M}{k} N\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq \frac{\Gamma_M - \gamma_M}{k} T\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq \frac{\Gamma_M - \gamma_M}{k} T(r, g) + S(r, g) \\ \Rightarrow \frac{(k+1)\gamma_M - \Gamma_M}{k} T(r, g) &\leq S(r, g) \\ &\Rightarrow T(r, g) = S(r, g), \end{aligned}$$

which is a contradiction. If $g(\zeta)$ is a non polynomial rational function, then we consider the following subcases:

Subcase I: $h_2 \neq 0$

For a non-zero h_2 , $M[g](\zeta) - h_2$ has at least one finite zero, see [17, Lemma 2.6, p.6]. This is a contradiction to the fact that $M[g](\zeta) \neq h_2$.

Subcase II: $h_2 = 0$

Since $M[g](\zeta) \neq h_2 = 0$, so we deduce

$$(2.16) \quad g(\zeta) = \frac{A}{(\zeta - b_1)^l},$$

where $A \neq 0$ is a constant and $l \geq 1$.

Now,

$$M[g](\zeta) = \frac{A^{\gamma_M}}{(\zeta - b_1)^{(l-1)\gamma_M + \Gamma_M}}.$$

So,

$$(2.17) \quad (M[g](\zeta))' = \frac{B}{(\zeta - b_1)^{(l-1)\gamma_M + \Gamma_M + 1}}.$$

Since, $M[g](a) = h_1$, we have

$$\begin{aligned} M[g](\zeta) - h_1 &= \frac{A^{\gamma_M}}{(\zeta - b_1)^{(l-1)\gamma_M + \Gamma_M}} - h_1 \\ &= C \frac{(\zeta - a)^{(l-1)\gamma_M + \Gamma_M}}{(\zeta - b_1)^{(l-1)\gamma_M + \Gamma_M}}, \end{aligned}$$

where $C \neq 0$ is a constant.

Therefore,

$$(2.18) \quad \begin{aligned} (M[g](\zeta))' &= (M[g](\zeta) - h_1)' \\ &= \frac{(\zeta - a)^{(l-1)\gamma_M + \Gamma_M - 1}}{(\zeta - b_1)^{(l-1)\gamma_M + \Gamma_M + 1}} R(\zeta), \end{aligned}$$

where $R(\zeta)$ is a polynomial. On comparing (2.17) and (2.18), we have

$$\begin{aligned} (l-1)\gamma_M + \Gamma_M - 1 + \deg(R(\zeta)) &= 0 \\ \Rightarrow (l-1)\gamma_M + \Gamma_M - 1 &\leq 0 \\ \Rightarrow (k-1)\gamma_M + \Gamma_M - 1 &\leq 0, \end{aligned}$$

which is a contradiction.

If $g(\zeta)$ is a non constant polynomial, then the polynomial $M[g](\zeta)$ cannot avoid h_1 and h_2 and so $P_w \circ M[g](\zeta) - \alpha(0)$ has at least two distinct zeros, which is a contradiction.

Case II: Suppose that $P_w \circ M[g](\zeta) - \alpha(0)$ has no zero. Then $M[g](\zeta) \neq h_1, h_2$. By proceeding the same way as in case I, we get a contradiction.

Thus the claim hold and hence there exist ζ_0 and $\tilde{\zeta}_0$ such that

$$P_w \circ M[g](\zeta_0) - \alpha(0) = 0$$

and

$$P_w \circ M[g](\tilde{\zeta}_0) - \alpha(0) = 0.$$

Consider $N_1 = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < \delta\}$ and $N_2 = \{\zeta \in \mathbb{C} : |\zeta - \tilde{\zeta}_0| < \delta\}$, where $\delta(> 0)$ is small enough number such that $N_1 \cap N_2 = \emptyset$ and $P_w \circ M[g](\zeta) - \alpha(0)$ has no other zeros

in $N_1 \cup N_2$ except for ζ_0 and $\tilde{\zeta}_0$. By Hurwitz theorem, there exist points $\zeta_j \rightarrow \zeta_0$ and $\tilde{\zeta}_j \rightarrow \tilde{\zeta}_0$ such that, for sufficiently large j , we have

$$P_w \circ M[f_j](z_j + \rho_j \zeta_j) - \alpha(z_j + \rho_j \zeta_j) = 0,$$

and

$$P_w \circ M[f_j](z_j + \rho_j \tilde{\zeta}_j) - \alpha(z_j + \rho_j \tilde{\zeta}_j) = 0.$$

Since, $P_w \circ M[f](z)$ and $P_w \circ M[g](z)$ share $\alpha(z)$ IM in D , we see for any integer i

$$P_w \circ M[f_i](z_j + \rho_j \zeta_j) - \alpha(z_j + \rho_j \zeta_j) = 0,$$

and

$$P_w \circ M[f_i](z_j + \rho_j \tilde{\zeta}_j) - \alpha(z_j + \rho_j \tilde{\zeta}_j) = 0.$$

For a fixed i , taking $j \rightarrow \infty$, we have

$$P_w \circ M[f_i](0) - \alpha(0) = 0.$$

Since, the zeros of $P_w \circ M[f_i](\zeta) - \alpha(\zeta)$ have no accumulation point except for finitely many f_i , so for sufficiently large j , we have

$$\begin{aligned} z_j + \rho_j \zeta_j &= 0, \quad z_j + \rho_j \tilde{\zeta}_j = 0 \\ \text{or } \zeta_j &= \frac{z_j}{\rho_j}, \quad \tilde{\zeta}_j = \frac{z_j}{\rho_j} \end{aligned}$$

which is a contradiction to the fact that $\zeta_j \in N_1$, $\tilde{\zeta}_j \in N_2$ and $N_1 \cap N_2 = \emptyset$. Hence \mathcal{F} is normal in D . ■

Proof of Corollary 1.5. Let $S = \{a_l(z) : l = 1, 2, \dots, n\}$ be the set of holomorphic functions on a domain D with $n \geq 2$. Then consider $P(z, w) := (w - a_1(z))(w - a_2(z)) \cdots (w - a_n(z))$ and $\alpha(z) \equiv 0$. Clearly, for every $f_i, f_j \in \mathcal{F}$, $P_w \circ M[f_i](z)$ and $P_w \circ M[f_j](z)$ share $\alpha(z)$ IM. Hence, by applying Theorem 1.2, F is normal in D . ■

Proof of Theorem 1.6. Suppose on contrary that \mathcal{G} is not normal at the origin. We consider the following cases:

Case I: $\alpha(0) \neq 0$

By Zalcman's lemma [16, p.216], we can find a sequence $\{f_j\}$ in \mathcal{G} , a sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ such that

$$g_j(\zeta) = \rho_j^{-\beta} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a non constant meromorphic function $g(\zeta)$ on \mathbb{C} having bounded spherical derivative.

We take $\beta = \frac{\Gamma_M}{\gamma_M} - 1$.

Hence

$$M[f_j](z_j + \rho_j \zeta) - \alpha(z_j + \rho_j \zeta) \rightarrow M[g](\zeta) - \alpha(0).$$

Clearly, $M[g](\zeta) \not\equiv \alpha(0)$. Thus by Lemma 2.1 and Lemma 2.2, $M[g](\zeta) - \alpha(0)$ has at least two distinct zeros. Suppose that ζ_0 and $\tilde{\zeta}_0$ be two distinct zeros of $M[g](\zeta) - \alpha(0)$. Consider $N_1 = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < \delta\}$ and $N_2 = \{\zeta \in \mathbb{C} : |\zeta - \tilde{\zeta}_0| < \delta\}$, where $\delta(> 0)$ is small enough number such that $N_1 \cap N_2 = \emptyset$ and $M[g](\zeta) - \alpha(0)$ has no other zeros in

$N_1 \cup N_2$ except for ζ_0 and $\tilde{\zeta}_0$. By Hurwitz theorem, there exist points $\zeta_j \rightarrow \zeta_0$ and $\tilde{\zeta}_j \rightarrow \tilde{\zeta}_0$ such that, for sufficiently large j , we have

$$M[f_j](z_j + \rho_j \zeta_j) - \alpha(z_j + \rho_j \zeta_j) = 0,$$

$$M[f_j](z_j + \rho_j \tilde{\zeta}_j) - \alpha(z_j + \rho_j \tilde{\zeta}_j) = 0.$$

Since, $M[f](z)$ and $M[g](z)$ share $\alpha(z)$ IM in D , we see for any integer i

$$M[f_i](z_j + \rho_j \zeta_j) - \alpha(z_j + \rho_j \zeta_j) = 0,$$

$$M[f_i](z_j + \rho_j \tilde{\zeta}_j) - \alpha(z_j + \rho_j \tilde{\zeta}_j) = 0.$$

For a fixed i , taking $j \rightarrow \infty$, we have

$$M[f_i](0) - \alpha(0) = 0.$$

Since, the zeros of $M[f_i](\zeta) - \alpha(0)$ have no accumulation point, so for sufficiently large j , we have

$$\begin{aligned} z_j + \rho_j \zeta_j &= 0, \quad z_j + \rho_j \tilde{\zeta}_j = 0 \\ \text{or } \zeta_j &= \frac{z_j}{\rho_j}, \quad \tilde{\zeta}_j = \frac{z_j}{\rho_j} \end{aligned}$$

which is a contradiction to the fact that $\zeta_j \in N_1$, $\tilde{\zeta}_j \in N_2$ and $N_1 \cap N_2 = \emptyset$.

Case II: $\alpha(0) = 0$

We assume $\alpha(z) = z^p \alpha_1(z)$, where p is a positive integer and $\alpha_1(0) \neq 0$. We may take $\alpha_1(0) = 1$. By Zalcman's lemma [16, p.216], we can find a sequence $\{f_j\}$ in \mathcal{G} , a sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow z_0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ such that

$$g_j(\zeta) = \rho_j^{-\eta} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a non constant meromorphic function $g(\zeta)$ on \mathbb{C} having bounded spherical derivative.

We take $\eta = \frac{\Gamma_M - \gamma_M + p}{\gamma_M}$

Now, we consider the following subcases:

Subcase I: Suppose there exist a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we take $\frac{z_j}{\rho_j}$ itself, such that $\frac{z_j}{\rho_j} \rightarrow \infty$ as $j \rightarrow \infty$.

Let $\Psi := \{H_j(\zeta) = z_j^{-\eta} f_j(z_j + z_j \zeta), \forall \{f_j\} \subset \mathcal{G}\}$. Thus by the given condition, we have

$$H_j^{n_0}(\zeta)(H_j')^{n_1}(\zeta) \cdots (H_j^{(k)})^{n_k}(\zeta) = (1 + \zeta)^p \alpha_1(z_j + z_j \zeta)$$

$$\Leftrightarrow G_j^{m_0}(\zeta)(G_j')^{n_1}(\zeta) \cdots (G_j^{(k)})^{n_k}(\zeta) = (1 + \zeta)^p \alpha_1(z_j + z_j \zeta),$$

where $G_j(\zeta) = z_j^{-\eta} g_j(z_j + z_j \zeta)$ for every $\{g_j\} \subset \mathcal{G}$. Thus, by case I, Ψ is normal in \mathbb{D} and hence there exist a subsequence of $\{H_j\}$ in Ψ , we may take $\{H_j\}$ itself, such that H_j converges spherically uniformly to H (say) on \mathbb{D} . We claim $H(0) = 0$. Suppose $H(0) \neq 0$, then we have

$$g_j(\zeta) = \rho_j^{-\eta} f_j(z_j + \rho_j \zeta) = \left(\frac{z_j}{\rho_j}\right)^{\eta} H_j\left(\frac{\rho_j}{z_j} \zeta\right)$$

converges locally uniformly to ∞ on \mathbb{C} . This implies that $g(\zeta) \equiv \infty$, which is a contradiction. Thus the claim holds and hence $H'(0) \neq \infty$. Now, for any $\zeta \in \mathbb{C}$, we have

$$g_j'(\zeta) = \rho_j^{-\eta+1} f_j'(z_j + \rho_j \zeta)$$

$$= \left(\frac{\rho_j}{z_j} \right)^{-\eta+1} H'_j \left(\frac{\rho_j}{z_j} \zeta \right) \xrightarrow{X} 0$$

on \mathbb{C} . Thus $g'(\zeta) \equiv 0$. This implies that g is a constant, which is a contradiction.

Subcase II: Suppose there exist a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we take $\frac{z_j}{\rho_j}$ itself, such that $\frac{z_j}{\rho_j} \rightarrow c$ as $j \rightarrow \infty$, where c is a finite number.

Let $\phi_j(\zeta) = \rho_j^{-\eta} f_j(\rho_j \zeta)$. Then

$$\phi_j(\zeta) = g_j \left(\zeta - \frac{z_j}{\rho_j} \right) \rightarrow g(\zeta - c) := \phi(\zeta).$$

Thus

$$\begin{aligned} & \phi_j^{n_0}(\zeta)(\phi'_j)^{n_1}(\zeta) \cdots (\phi_j^{(k)})^{n_k}(\zeta) - \zeta^p \\ &= g^{n_0} \left(\zeta - \frac{z_j}{\rho_j} \right) (g')^{n_1} \left(\zeta - \frac{z_j}{\rho_j} \right) \cdots (g^{(k)})^{n_k} \left(\zeta - \frac{z_j}{\rho_j} \right) - \left(\zeta - \frac{z_j}{\rho_j} \right)^p \\ &\rightarrow g^{n_0}(\zeta - c)(g')^{n_1}(\zeta - c) \cdots (g^{(k)})^{n_k}(\zeta - c) - (\zeta - c)^p \\ &:= \phi^{n_0}(\zeta)(\phi')^{n_1}(\zeta) \cdots (\phi^{(k)})^{n_k}(\zeta) - \zeta^p \end{aligned}$$

Clearly, $\phi^{n_0}(\zeta)(\phi')^{n_1}(\zeta) \cdots (\phi^{(k)})^{n_k}(\zeta) - \zeta^p \not\equiv 0$. Thus by Lemma 2.1 and Lemma 2.2, $\phi^{n_0}(\zeta)(\phi')^{n_1}(\zeta) \cdots (\phi^{(k)})^{n_k}(\zeta) - \zeta^p$ has at least two distinct zeros. Thus by proceeding the same way as in Case I, we get a contradiction. Hence \mathcal{G} is normal in D . \blacksquare

Proof of Theorem 1.7. Suppose \mathcal{F} is not normal at the origin. Then by Zalcman's lemma [16, p.216], we can find a sequence $\{f_j\}$ in \mathcal{F} , a sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \rightarrow 0$ such that

$$g_j(\zeta) = \rho_j^{-\beta} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a non constant meromorphic function $g(\zeta)$ on \mathbb{C} having bounded spherical derivative.

We take $\beta = \frac{\Gamma_M}{\gamma_M} - 1$.

Hence,

$$(2.19) \quad P_w \circ M[f_j](z_j + \rho_j \zeta) - \alpha(z_j + \rho_j \zeta) \rightarrow P_w \circ M[g](\zeta) - \alpha(0).$$

Clearly, $P_w \circ M[g](\zeta) \not\equiv \alpha(0)$. We claim that $P_w \circ M[g](\zeta) - \alpha(0)$ has at most one zero. Suppose that $P_w \circ M[g](\zeta) - \alpha(0)$ has two distinct zeros say ζ_1 and ζ_2 . Then by (2.19) and Hurwitz theorem, there exist points $\zeta_{j1} \rightarrow \zeta_1$ and $\zeta_{j2} \rightarrow \zeta_2$ such that

$$P_w \circ M[f_j](z_j + \rho_j \zeta_{j1}) - \alpha(z_j + \rho_j \zeta_{j1}) = 0$$

and

$$P_w \circ M[f_j](z_j + \rho_j \zeta_{j2}) - \alpha(z_j + \rho_j \zeta_{j2}) = 0,$$

for sufficiently large j . Since, $z_j + \rho_j \zeta_{ji} \rightarrow 0$ ($i = 1, 2$) and $P_w \circ M[f_j](z_j + \rho_j \zeta) - \alpha(z_j + \rho_j \zeta)$ has at most one zero, we get a contradiction. Thus the claim holds. But as shown in the proof of Theorem 1.2, $P_w \circ M[g](\zeta) - \alpha(0)$ has at least two distinct zeros and this contradicts our claim. Hence \mathcal{F} is normal in D . \blacksquare

Proof of Theorem 1.8. Suppose \mathcal{F} is not normal at the origin. Then as in the proof of the Theorem 1.7, we find that

$$P_w \circ M[f_j](z_j + \rho_j \zeta) - \alpha(z_j + \rho_j \zeta) \rightarrow P_w \circ M[g](\zeta) - \alpha(0).$$

Clearly, $P_w \circ M[g](\zeta) \neq \alpha(0)$. As shown in the proof of Theorem 1.2, $P_w \circ M[g](\zeta) - \alpha(0)$ has at least one zero say ζ_0 and hence $g(\zeta_0) \neq \infty$. By Hurwitz theorem, for sufficiently large j , there exist a sequence $\zeta_j \rightarrow \zeta_0$ as $j \rightarrow \infty$ such that

$$P_w \circ M[f_j](z_j + \rho_j \zeta_j) - \alpha(z_j + \rho_j \zeta_j) = 0.$$

Thus by the given condition, $P_w \circ M[f](z) = \alpha(z)$ implies $|f(z)| \geq M$, we have

$$\begin{aligned} |g_j(\zeta_j)| &= \rho_j^{-\beta} |f_j(z_j + \rho_j \zeta_j)| \\ &\geq \rho_j^{-\beta} M. \end{aligned}$$

Since $g(\zeta_0) \neq \infty$ in some neighborhood of ζ_0 say N_{ζ_0} , it follows that for sufficiently large values of j , $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in N_{ζ_0} . Thus for $\epsilon > 0$ and for every $\zeta \in N_{\zeta_0}$, we have

$$|g_j(\zeta) - g(\zeta)| < \epsilon.$$

Therefore, for sufficiently large values of j , we have

$$|g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| \geq \rho_j^{-\beta} M - \epsilon,$$

which is a contradiction to the fact that ζ_0 is not a pole of $g(\zeta)$. Hence \mathcal{F} is normal in D . ■

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU-180 006, INDIA
E-mail address: virendersingh2323@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU-180 006, INDIA
E-mail address: kscharak7@rediffmail.com